Micromechanics of Unsaturated Granular Media

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Abstract: The homogenization method is used to study the properties of the strength criterion of a granular material in the dry, saturated, and unsaturated situations. Adopting a periodic description of the granular material at the microscopic level, the main features of the up-scaling technique are recalled. Then, a general definition of the strength criterion at the macroscopic scale is given in the framework of yield design homogenization theory. This approach makes it possible to find again classical properties of the strength criterion as a function of the properties of the granular material at the microscopic level. Furthermore, some experiments are presented in order to assess the accuracy of the model used to describe the unsaturated granular medium at the microscopic level.

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Introduction

It is well known that the addition of liquid within a dry granular medium, even in small quantity, can modify in a significant way its behavior when compared to the dry state. Thus, whereas a dry sand does not have any cohesion, it is enough to add a small quantity of water to observe a tensile strength. This increase of strength capacity is classically explained by the formation of liquid bridges located at the contact between grains (Marshall and Holmes 1988; Fredlund and Rahardjo 1993). It is this property which allows us to build sand castle.

In this paper, the yield design homogenization method is used to characterize the macroscopic yield strength properties of unsaturated granular media from the description of the physics and of the morphology at the microscopic level.

To begin with, the main features of the method are presented in the cases of dry and fully saturated granular material. Then, the unsaturated situation is studied. Restricting ourselves to the situation where the porous space is filled by two fluids, it is shown how the interface phenomena located at the boundaries between volumic phases have to be taken into account in the up-scaling approach. These results at hand, a definition of the strength criterion of the unsaturated granular material is given. This definition is successively applied to the situation where the wetting fluid phase occupies a continuous domain containing bubbles of the nonwetting fluid phase (high-saturation ratio) and to the situation where the wetting fluid phase occupies separate domains located around the contact point (low-saturation ratio). In both cases, it is shown that the determination of the macroscopic strength criterion simplifies to determine the macroscopic strength criterion for the same granular material without surface tension effect, providing that the local strength criterion be replaced by the appropriate one.

Of course, the validity of such an up-scaling approach relies upon the relevancy of the description of the medium at the microscopic scale. To answer this question, some results from an experimental approach to the behavior of a liquid bridge embedded between two solid surfaces are briefly reported and, then, compared to different theoretical approaches.

Dry Granular Materials

Whereas several works devoted to the behavior of dry granular materials by means of up-scaling methods can be found in the literature (Weber 1966; Christofferson et al. 1981; Cambou and Sidoroff 1985; Hicher and Rahma 1994; Caillerie 1995; Zhuang et al. 1995; Emeriault and Cambou 1996; Sab 1996; Cambou et al. 2000), not so much attention seems to have been paid to the determination of the strength capacity of this kind of material. Furthermore, if a consensus has been drawn concerning the relation linking internal forces at the microscopic scale to the internal forces at the macroscopic scale, the situation did not counter so clear as regards the kinematical variables. In order to overcome these difficulties, the main features of the homogenization method suited to periodic dry granular material are now presented.

Homogenization for Periodic Dry Granular Materials

Consider a three-dimensional structure occupying a domain Ω made up of a periodic granular material. Let $\mathcal{C}$ denote a representative unit cell which may be regarded as the smallest representative volume of material. Thanks to the periodicity, it is always possible to select as a representative cell a domain whose boundary does not intersect any grain.

It is assumed that the contact between particles reduces to a single geometrical point located at the boundary of each contacting grain and that the force exerted by one grain to another can be accurately described by a single vector force.

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It is well known that, when appropriate, the periodical homogenization method defines the link between the microscopic and macroscopic properties of the medium under consideration from the response of the representative cell to a mechanical loading where the macroscopic stress tensor appears as a loading parameter (Auriault and Sanchez-Palencia 1977; Suquet 1982; de Buhan 1986). More precisely, the boundary conditions over the representative cell have to correspond to a loading mode depending linearly on the six (in three dimensions) independent components of the Cauchy stress tensor $\mathbf{\Sigma}$ defined so that the macroscopic Cauchy stress tensor appears as the average over the representative cell of the microscopic internal forces. Moreover, the macroscopic kinematic parameters [the virtual strain rate in the framework of yield design theory (Salençon 1990), or the strain tensor in the framework of deformable solid theory (Suquet 1982)] associated to the macroscopic internal force $\mathbf{\Sigma}$ are also related to their microscopic counterpart by an average relation so as to check the Hill lemma, which ensures the equality of the power (or the work) of the internal forces in the kinematic variables whether this quantity is computed at the microscopic level or the macroscopic level.

This definition suits easily to the discrete model considered here as follows.

Consider the representative cell $\mathcal{C}$ made up of N_g grains labeled with suffixes k. The contact points are labeled by c. Here, the set of the contact points consists of all the points located on the inner side of the representative cell $\mathcal{C}$ and of all the contact points located at the cell's boundary. N_c denotes the number of points belonging to this set. At contact c, k_1^c and k_2^c denote, respectively, the suffix of the grain contacting each other while $\underline{\mathbf{F}}^c$ is the force exerted by grain number k_2^c on grain k_1^c. This definition does not hold for the contact located at the boundary of $\mathcal{C}$. In this case, $\underline{\mathbf{F}}^c$ is the force exerted by the outside on grain k_1^c.

It is useful for the sequel to define for each grain belonging to the representative unit cell $\mathcal{C}$ a "center" denoted by $\underline{\boldsymbol{\zeta}}_k$ ($k=1,N_g$). The geometric position of this center can be chosen arbitrarily without needing to refer to a particular geometrical or mechanical property.

First of all, a microscopic contact forces field $(\underline{\mathbf{F}}^c)_{c=1,N_c}$ is said to be statically admissible (SA) with the symmetric stress tensor $\mathbf{\Sigma}$ if the contact forces take opposite values on opposite sides of the elementary cell, if $(\underline{\mathbf{F}}^c)_{c=1,N_c}$ comply with the equilibrium requirement (force and moment) for each grain, and if $\mathbf{\Sigma}$ is linked to the contact forces by the linear equation

$$(\forall \underline{\mathbf{F}}^c \quad \mathrm{SA})$$
$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\left(\sum_{c=1}^{N_b}\underline{\mathbf{x}}^c\otimes\underline{\mathbf{F}}^c\right)=\frac{1}{|\mathcal{C}|}\left(\sum_{c=1}^{N_c}\underline{\boldsymbol{\ell}}^c\otimes\underline{\mathbf{F}}^c\right) \tag{1}$$

where $\otimes$=tensorial product; N_b=number of contact points located on the cell's boundary $\underline{\boldsymbol{\ell}}^c=\underline{\boldsymbol{\zeta}}_{k_2^c}-\underline{\boldsymbol{\zeta}}_{k_1^c}$ if $\underline{\mathbf{x}}^c$ is located on the inner side of $\mathcal{C}$; and $\underline{\boldsymbol{\ell}}^c=\underline{\mathbf{x}}_c-\underline{\boldsymbol{\zeta}}_{k_1^c}$ if $\underline{\mathbf{x}}_c$ is located on the boundary of $\mathcal{C}$. The first equality of Eq. (1) corresponds to the classical relation linking the average of a no-body force stress tensor computed over a domain to the values of the surface traction forces applied on the boundary of the domain which in the continuous case follows from

$$(\operatorname{div}\boldsymbol{\sigma}=0 \quad \text{in} \quad \mathcal{C})\Rightarrow|\mathcal{C}|\langle\boldsymbol{\sigma}\rangle$$
$$=\int_{\mathcal{C}}\boldsymbol{\sigma}\,d\Omega=\int_{\partial\mathcal{C}}\underline{\mathbf{x}}\otimes\boldsymbol{\sigma}\cdot\underline{\mathbf{n}}\,dS \tag{2}$$

where $\underline{\mathbf{n}}$=outward unit normal to $\mathcal{C}$. Taking into account that surface tractions reduce to forces applied on single points in the dry granular situation yields the first equality (Drescher and de Josselin de Jong 1972). The second equality is obtained by adding the contribution of each grain to the averaged stress

$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\sum_{k=1}^{N_g}\left(\sum_{c=1}^{N_c^k}\underline{\mathbf{x}}_k^c\otimes\underline{\mathbf{F}}_k^c\right)=\frac{1}{|\mathcal{C}|}\sum_{k=1}^{N_g}\left(\sum_{c=1}^{N_c^k}(\underline{\mathbf{x}}_k^c-\underline{\boldsymbol{\zeta}}_k)\otimes\underline{\mathbf{F}}_k^c\right)$$
$$+\frac{1}{|\mathcal{C}|}\sum_{k=1}^{N_g}\underline{\boldsymbol{\zeta}}_k\otimes\left(\sum_{c=1}^{N_c^k}\underline{\mathbf{F}}_k^c\right) \tag{3}$$

where N_c^k=number of contact point for the grain number k and $\underline{\mathbf{F}}_k^c$=force applied to the grain number k at contact c. Using the balance equation for each grain allows to eliminate the last term of Eq. (3). Then, the second equality of Eq. (1) is obtained from Eq. (3) by using the fact that the forces applied to the grains contacting one to the other are opposite together with the definition of the vector $\underline{\boldsymbol{\ell}}^c$ given above. Furthermore, it is easily shown that for any statically admissible contact forces field, the whole cell complies with the moment equilibrium requirement, which implies the symmetry of the macroscopic Cauchy stress tensor $\mathbf{\Sigma}$ defined by relation (1).

Similarly, a virtual velocity fields of the form $\underline{\hat{\mathbf{u}}}=\hat{\mathbf{D}}\cdot\underline{\mathbf{x}}+\underline{\hat{\mathbf{v}}}$, where $\hat{\mathbf{D}}$ is a symmetric second-order tensor and $\underline{\hat{\mathbf{v}}}$ is a periodic velocity field is said to be kinematically admissible (KA) with $\hat{\mathbf{D}}$. Because the statics of granular media is described here by a discrete model, it is sufficient to consider velocity fields whose restriction to each grain making up the elementary cell is a rigid body motion.

Then using the principle of virtual work for any contact force fields statically admissible with $\mathbf{\Sigma}$ and for any velocity fields kinematically admissible with $\hat{\mathbf{D}}$ yields the Hill theorem suited to dry granular materials

$$\begin{pmatrix}\forall\underline{\mathbf{F}}^c & \mathrm{SA} & \text{with } \mathbf{\Sigma}\\ \forall\underline{\hat{u}} & \mathrm{KA} & \text{with } \hat{\mathbf{D}}\end{pmatrix}\quad \mathbf{\Sigma}:\hat{\mathbf{D}}=\frac{1}{|\mathcal{C}|}\left(\sum_{c=1}^{N_c}[\![\underline{\hat{u}}^c]\!]\cdot\underline{\mathbf{F}}^c\right) \tag{4}$$

where $[\![\underline{\hat{\boldsymbol{u}}}^c]\!]=\underline{\hat{\boldsymbol{u}}}(\underline{\mathbf{x}}^c\in g_{k_2^c})-\underline{\hat{\boldsymbol{u}}}(\underline{\mathbf{x}}^c\in g_{k_1^c})$=velocity jump across the contact interface at point $\underline{\mathbf{x}}^c$ and g^k=geometrical domain occupied by the grain number k. Some special attention must be paid to the contact points located at the boundary of the unit cell when computing Eq. (4).

The first possibility, which is also the simplest one, is to consider that the virtual velocity of a contact point located on the boundary of the cell is equal to the virtual velocity of the particle belonging to the grain k_1^c. The second possibility is to consider the contact points located at the boundary as the contact points located on the inner side of the unit representative cell and then to distinguish two particles located at the contact point $\underline{\mathbf{x}}_c$, one belonging to the grain number k_1^c, the second one belonging to the grain located apart from the cell. In that case, one has to consider discontinuities of the virtual velocity fields for all the contact points. Finally, according to the periodicity properties of the problem, it is also possible to consider virtual velocity fields such as velocity discontinuities can occur only for half of the points located on the cell boundary, these points being selected in order to prevent that two of them correspond by periodicity. The reasons why the last choice appears to us as the best can be easily explained. Indeed, even if Eq. (4) applies for any choice of the velocity provided that it complies with the kinematic requirement, it is obvious that when one has to use Hill's lemma in order to

J. Eng. Mech., 2002, 128(8): 856-863

perform an up-scaling approach, taking into account or not velocity jumps at the boundary contact points can produce different results. On one side, it is clear that considering continuous velocity fields across the boundary contact points does not allow us to take into account the behavior of these points in the up-scaling approach. On the other side, it seems inadequate to consider velocity jumps for all the points located on the boundary since this amounts to counting twice each of the boundary contact points and then to overweight these points in the average process. Finally, it is clear that taking into account just half of the cell's boundary contact points allows us to incorporate in the homogenization method the behavior associated with these points while avoiding to overweight their contribution.

One can summarize the reasoning which has just been presented by saying that in spite of the fact that they occupy a volume of no measure, contacts between grains are material elements which strongly influence the behavior of the medium. Thus, a good representative cell should include both the grains and the contact points necessary to reconstruct the whole periodic structure.

Whereas the relationship (1) between microscopic and macroscopic is now classical (Weber 1966; Caillerie 1995; Cambou et al. 2000), the situation is not so clear for the relationship between the kinematic variables, especially in the random case (Sab 1996; Cambou et al. 2000). Here, one takes advantage of the periodicity property to define the macroscopic rate strain tensor from the average strain tensor of the elementary cell. This avoids the classical difficulties encountered when one tries to generalize the classical average relation well suited to the continuum media (Cambou et al. 2000). In our opinion, the simplest way to proceed for random materials is to use a Hashin-like boundary condition in order to define the mechanical loading over the representative elementary volume (Bourada-Benyamina 1999). Different solutions were proposed to define this relationship (Sab 1996; Cambou et al. 2000). It can be noted that in most of these works, an extension of the microscopic strain definition from the contact point velocity jump to the whole domain occupied by the representative elementary volume is used.

Strength Criterion for Dry Granular Material

According to the yield design homogenization theory (de Buhan 1986), the determination of the macroscopic strength criterion of the above-described dry granular material amounts to solving a yield design boundary problem defined over the representative elementary cell. More precisely, introducing for each contact point $\underline{\mathbf{x}}_c$ belonging to $\mathcal{C}$ the convex domain $G^{\text{int}}(\underline{\mathbf{x}}_c)$ characterizing the strength capacities of the contact interface, one may define the macroscopic strength domain, denoted by $\mathcal{G}$, as the set of macroscopic stress tensor $\mathbf{\Sigma}$ such that there exists a force field $(\underline{\mathbf{F}}^c)_{c=1,N_c}$ statically admissible with $\mathbf{\Sigma}$ and compatible with the microscopic strength condition

$$\mathbf{\Sigma}\in\mathcal{G}\Leftrightarrow\exists(\underline{\mathbf{F}}^c)\ \text{ such as }\ \begin{cases}(\underline{\mathbf{F}}^c)\ \text{ SA with }\ \mathbf{\Sigma}\\ \forall c\,\underline{\mathbf{F}}^c\in G^{\text{int}}(\underline{\mathbf{x}}^c)\end{cases}\tag{5}$$

The definition (5) does not take into account the strength capacities of the grain matter. This implies that it is assumed that the constituent material of the grains is infinitely resistant. As a matter of fact it is not possible to take into account the strength capacities of the grain because of the singularity of the grain's stress tensor at the contact points (Bourada-Benyamina 1999).

A particularly important case from a practical point of view is that when dry friction and unilateral contact characterize the strength capacities of the contact interface. Then $G^{\text{int}}(\underline{\mathbf{x}}^c)$ writes

$$G^{\text{int}}(\underline{\mathbf{x}}^c)=\{\underline{\mathbf{F}}_c\ \text{ such as }\ |\underline{\mathbf{T}}_c|-N_c\tan\varphi\leqslant 0\}\tag{6}$$

where $N_c=\underline{\mathbf{F}}_c\cdot\underline{\mathbf{n}}$, and $\underline{\mathbf{T}}_c=\underline{\mathbf{F}}_c-N_c\underline{\mathbf{n}}\cdot\underline{\mathbf{n}}$=outward unit normal to grain k_1^c at point $\underline{\mathbf{x}}_c$ and φ denotes the friction angle. As the set of the statically admissible forces with any value of $\mathbf{\Sigma}$ is a vector space and the set of all admissible forces is a cone with apex the naught vector, it is obvious that the macroscopic strength criterion defined by Eq. (5) as to the intersection of this two sets, is also a cone with apex the origin in the space of the symmetric second-order tensor.

From a practical point of view, it is possible to calculate the set $\mathcal{G}$ for a given geometry of the elementary cell directly from the definition (5) by means of a double projection algorithm (Bourada-Benyamina 1999) or by means of a more classical nonlinear optimization procedure.

Fully Saturated Granular Material

We now examine the situation of the fully saturated granular material whose porous space is filled with a pressurized fluid. As for the dry case, the main features of periodic homogenization are recalled before the results are applied to define the macroscopic strength criterion of such media.

Homogenization for Periodic Saturated Granular Media

As it is classical in the framework of homogenization methods applied to determine macroscopic properties connected with the microscopic behavior of the solid matrix, the pressure p is regarded as uniform at the scale of the representative elementary cell (Auriault and Sanchez-Palencia 1977). First of all, it must be observed that, thanks to the fact that contact interfaces between grains were supposed to reduce to single points, the whole force and the whole moment exerted by the uniform pressure p on each grain is naught. Then, performing exactly the same kind of reasoning as for the dry situation, a microscopic internal force field $((\underline{\mathbf{F}}^c)_{c=1,N_c},p)$ is said to be statically admissible if the contact forces take opposite values on opposite sides of the elementary cell and if $(\underline{\mathbf{F}}^c)_{c=1,N_c}$ comply with the equilibrium requirement (force and moment) for each grain.

Then, the value of the macroscopic stress tensor $\mathbf{\Sigma}$ associated to the contact forces and to pressure p statically admissible is the average of the internal forces over the geometric domain $\mathcal{C}$. Performing exactly the same kind of reasoning that for the dry case yields

$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\left(\sum_{c=1}^{N_c}\underline{\ell}^c\otimes\underline{\mathbf{F}}^c\right)-p\,\boldsymbol{\delta}\tag{7}$$

where $\boldsymbol{\delta}$ denotes the second-order identity tensor.

Strength Criterion for Saturated Granular Material

Then, the definition of the macroscopic strength criterion suited to the saturated case writes

$$\mathbf{\Sigma}\in\mathcal{G}(p)\Leftrightarrow\exists(\underline{\mathbf{F}}^c)\ \text{ such as }\ \begin{cases}[(\underline{\mathbf{F}}^c),p]\ \text{ SA with }\ \mathbf{\Sigma}\\ \forall c\,\underline{\mathbf{F}}^c\in G^{\text{int}}(\underline{\mathbf{x}}^c)\end{cases}\tag{8}$$

As in the dry situation, the determination of the macroscopic strength criterion of a periodic saturated granular material reduces

J. Eng. Mech., 2002, 128(8): 856-863

to solving a discrete yield design boundary-value problem defined over the grains of the representative elementary cell.

Assuming that the microscopic strength criterion is affected neither by the presence of fluid nor by the value of the pressure p, it is easily shown from definition (8) that

$$[\boldsymbol{\Sigma}\in\mathcal{G}(0)]\Leftrightarrow[\forall p\in\mathbb{R},\boldsymbol{\Sigma}-p\boldsymbol{\delta}\in\mathcal{G}(p)] \quad (9)$$

Property (9) ensures that the macroscopic strength condition relates solely to the effective stress $\boldsymbol{\Sigma}+p\boldsymbol{\delta}$ (the "+" sign comes from the fact that the convention of positive traction stress is used) and then can be identified from both dry or saturated experiments.

If the microscopic strength condition depends upon the nature of the fluid saturing the porous space, but is unaffected by the value of p, the effective stress principle remains valid but the identification of the strength set $\mathcal{G}(0)$ has to be done for each fluid saturating the porous space. The results presented here for the saturated case have been initially provided in (de Buhan and Dormieux 1996).

Unsaturated Granular Material

We now examine the case of a granular material which porous space if filled by two immiscible fluids, namely, a liquid and a gas. It is then necessary to take into account the capillary effects.

Statics of Unsaturated Granular Media

The representative elementary cell Ω is now made up of the grains, a liquid phase and a gaseous phase which, respectively, occupy the domain $\Omega^s=\cup g^k$, Ω^ℓ and $\Omega^g\cdot\omega^{\alpha\beta}$ denotes the interface between the α and β phases. Of course, the shape of this interface has to comply with the periodicity property. At the microscopic scale, the internal forces are described by the fluid pressure p^α $(\alpha=\ell,g)$ in the fluid phases and the contact forces between grains. The capillary effects introduce internal forces of the membrane type located in the interfaces between phases. The latter are represented by a tensor field of surface tension $\gamma^{\alpha\beta}\delta_{T_\omega}$ in the surface $\omega^{\alpha\beta}$, where $\boldsymbol{\delta}_{T_\omega}$ denotes the unit tensor to the tangent plane to surface ω and $\gamma^{\alpha\beta}$ the surface tension in the $\alpha\beta$ interface, with $(\alpha\beta)=(s\ell)$, (sg), and (ℓg).

Thus, to be statically admissible, the microscopic stress field has to comply with the momentum balance equation div$\boldsymbol{\sigma}=0$. In particular, this implies that p^ℓ and p^g are uniform in Ω^g and Ω^ℓ, respectively. The equilibrium requirement for grains has now to be reconsidered in order to incorporate the interface effects. Taking the capillary effects into account, the classical condition of continuity of the stress vector at the interfaces $\omega^{s\alpha}$ is replaced by the following conditions:

$$\boldsymbol{\sigma}\cdot\underline{\mathbf{n}}=-p^\alpha\underline{\mathbf{n}}-\gamma^{s\alpha}(\boldsymbol{\delta}_{T_\omega}:\mathbf{b})\underline{\mathbf{n}} \quad (\forall\underline{\mathbf{x}}\in\omega^{s\alpha})(\alpha=g,\ell) \quad (10)$$

where $\underline{\mathbf{n}}$ denotes the outer unit normal vector to g^k and $\mathbf{b}$=tensor of curvature. $\boldsymbol{\delta}_{T_\omega}:\mathbf{b}$ thus represents the mean curvature of $\omega^{s\alpha}$.

The corresponding condition at the interface $\omega^{g\ell}$ is classically referred to as Laplace law

$$(\forall\underline{\mathbf{x}}\in\omega^{\ell g}) \quad \gamma^{\ell g}\boldsymbol{\delta}_{T_\omega}:\mathbf{b}=p^\ell-p^g=-p^c \quad (11)$$

The momentum equation with no body force for each grain writes now

$$\sum_{c=1}^{N_c^k}\underline{\mathbf{F}}_k^c+\underline{\mathbf{F}}_k^{un}=0 \quad \text{and} \quad \sum_{c=1}^{N_c^k}\underline{\mathbf{x}}^c\wedge\underline{\mathbf{F}}_k^c+\underline{\mathbf{M}}_k^{un}=0 \quad (12)$$

It is recalled that N_k^c denotes the number of contact points for grain number k and $\underline{\mathbf{F}}_k^c$ is the contact force applied onto grain number k at point $\underline{\mathbf{x}}_k^c\cdot\underline{\mathbf{F}}_k^{un}$ (respectively, $\underline{\mathbf{M}}_k^{un}$) is the sum of all the forces (respectively, the moment) exerted by the interfaces onto the considered grain.

$$\begin{cases}\underline{\mathbf{F}}_k^{un}=\sum_{\alpha=\ell,g}\int_{\omega_k^{s\alpha}}-(p^\alpha+\gamma^{s\alpha}\boldsymbol{\delta}_{T_\omega}:\mathbf{b})\underline{\mathbf{n}}dS+\int_{\partial\omega_k^{s\ell g}}\gamma^{\ell g}\underline{\boldsymbol{\nu}}\ ds\\ \underline{\mathbf{M}}_k^{un}=\sum_{\alpha=\ell,g}\int_{\omega_k^{s\alpha}}-\underline{\mathbf{x}}\wedge(p^\alpha+\gamma^{s\alpha}\boldsymbol{\delta}_{T_\omega}:\mathbf{b})\underline{\mathbf{n}}dS+\int_{\partial\omega_k^{s\ell g}}\underline{\mathbf{x}}\wedge\gamma^{\ell g}\underline{\boldsymbol{\nu}}\ ds\end{cases} \quad (13)$$

with $\omega_k^{s\ell}=\partial g^k\cap\omega^{s\ell}$, $\omega_k^{sg}=\partial g^k\cap\omega^{sg}$ and $\partial\omega_k^{s\ell g}=\omega_k^{s\ell}\cap\omega_k^{sg}$.

The last term of Eqs. (13) corresponds to the force exerted by the liquid-gas interface onto the grain number k along the tripleline $\partial\omega_k^{s\ell g}=\omega_k^{s\ell}\cap\omega_k^{sg}\cdot\underline{\boldsymbol{\nu}}$ denotes the inward unit normal to $\omega^{\ell g}$ tangent to $\omega^{\ell g}$ along $\partial\omega^{s\ell g}$. As no line tension is taken into account in this work, the value of the wetting angle θ, which is the angle between, respectively, the liquid-gas and the solid-liquid tangent plan along the triple line, is defined by the classical Young relation

$$\gamma^{\ell g}\cos\theta+\gamma^{s\ell}=\gamma^{sg} \quad (14)$$

Homogenization for Periodic Unsaturated Granular Material

It is now possible to generalize the definition given above in order to incorporate the capillary effects into the average relationship linking microscopic internal forces to macroscopic ones. Considering a representative elementary cell of unsaturated granular material, the microscopic internal forces $((\underline{\mathbf{F}}^c)_{c=1,N_c},p^g,p^\ell)$ are said to be statically admissible with the macroscopic stress tensor $\boldsymbol{\Sigma}$ if p^ℓ and p^g are two uniform pressure fields complying with Laplace law (11), if the contact forces $(\underline{\mathbf{F}}^c)_{c=1,N_c}$ comply with the momentum balance Eq. (12) and take opposite values on opposite sides of the elementary cell and if the microscopic and macroscopic internal forces are linked together by the linear relation

$$\begin{aligned}\boldsymbol{\Sigma}=\frac{1}{|\mathcal{C}|}\Bigg[&\sum_{c=1}^{N_c}\underline{\boldsymbol{\ell}}^c\otimes\underline{\mathbf{F}}^c\\ &+\sum_{k=1}^{N_g}\left(\int_{\omega_k^{s\ell}}\underline{\tilde{\mathbf{x}}}_k\otimes p^c\underline{\mathbf{n}}\,dS+\int_{\partial\omega_k^{s\ell g}}\underline{\tilde{\mathbf{x}}}_k\otimes\gamma^{\ell g}\underline{\boldsymbol{\nu}}\,ds\right)\\ &+\int_{\omega^{\ell g}}\gamma^{\ell g}\boldsymbol{\delta}_{T_\omega}dS\Bigg]+n^\ell p^c\boldsymbol{\delta}-p^g\boldsymbol{\delta}\end{aligned} \quad (15)$$

where $\underline{\tilde{\mathbf{x}}}_k$=local coordinates system attached to the grain number k defined by $\underline{\tilde{\mathbf{x}}}_k=\underline{\mathbf{x}}-\underline{\boldsymbol{\zeta}}_k$. It is recalled that $\boldsymbol{\zeta}_k$ is the "center" of grain k used to defined the contact vectors $\underline{\boldsymbol{\ell}}_c$.

The generalized average relation (15) is derived from the average relation valid in the continuous case (Chateau and Dormieux 1995). Denoting by $\boldsymbol{\sigma}$ the stress tensor in the grains, the average relation writes for the unsaturated granular material

J. Eng. Mech., 2002, 128(8): 856-863

$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\left(\sum_{k=1}^{N_g}\int_{g^k}\boldsymbol{\sigma}\, d\Omega+\int_{\omega}\gamma\boldsymbol{\delta}_{T_\omega}dS\right)-n^\ell p^\ell\boldsymbol{\delta}-n^g p^g\boldsymbol{\delta} \tag{16}$$

Observing that the material system made up of one grain, its solid-liquid interface and its solid-gas interface is at rest with no body force, it is possible to compute its contribution to the macroscopic stress tensor as a function of the surface traction applied on its boundary by means of relation (2). Then, performing exactly the same calculations as for dry case yields Eq. (15).

Strength Criterion for Unsaturated Granular Material

The relation linking the statics of unsaturated granular material at the microscopic level to a macroscopic continuous model being now established, one has to perform exactly the same reasoning as for the dry case in order to define the macroscopic strength condition from the microscopic description of the unsaturated granular material. More precisely, the macroscopic strength criterion is defined by

$$\mathbf{\Sigma}\in\mathcal{G}(p^\ell,p^g)\Leftrightarrow\exists(\underline{\mathbf{F}}_c)_{c=1,N_c},p^\ell,p^g)\quad\text{such as}$$

$$\begin{cases}((\underline{\mathbf{F}}_c)_{c=1,N_c},p^\ell,p^g)\ \text{SA with}\ \mathbf{\Sigma}\\ \forall\underline{\mathbf{x}}_c\underline{\mathbf{F}}_c\in G_c\end{cases} \tag{17}$$

Two different situations are now examined: the first one corresponding to the high-saturation ratio while the second one corresponds to low-saturation ratio. Let us recall that the saturation ratio is defined as the ratio of the volume filled by liquid to the volume of the porous space. In both situations, it is assumed that the liquid phase wets the solid.

High-Saturation Ratio

Consider the case where the gas phase occupies small domains surrounded by the liquid phase. In this situation the solid-gas interface reduces to single points located on the boundary of the grains or to the empty set. Then, for each grain belonging to the representative elementary cell, the liquid-phase domain surrounds the domain g^k occupied by the grain so that $\omega_k^{s\ell}=\partial g^k$ and $\partial\omega_k^{s\ell g}=\varnothing$. Then, using Eq. (2) with $\boldsymbol{\sigma}=p^c\boldsymbol{\delta}$ and $\mathcal{C}=g^k$, Eq. (15) now writes

$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\left(\sum_{c=1}^{N_c}\underline{\ell}^c\otimes\underline{\mathbf{F}}^c+\int_{\omega^{\ell g}}\gamma^{\ell g}\delta_{T_\omega}dS\right)+(n^s+n^\ell)p^c\boldsymbol{\delta}-p^g\boldsymbol{\delta} \tag{18}$$

Furthermore, it is well known that a gas domain embedded in a liquid domain, the whole being at rest, occupies a spherical domain according to the Laplace equation. Then, using the same kind of reasoning that the one used in order to compute Eq. (7) yields the relation

$$\int_{\omega^{\ell g}}\gamma^{\ell g}\boldsymbol{\delta}_{T_\omega}dS=\int_{\omega^{\ell g}}\underline{\mathbf{x}}\otimes p^c\underline{\mathbf{n}}\, dS=|\Omega^g|p^c\boldsymbol{\delta} \tag{19}$$

where $\underline{\mathbf{n}}$ denotes the outward unit normal to the gas domain.

Then, introducing Eq. (19) in Eq. (18) together with the relation $p^c=p^g-p^\ell$ implies that

$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\left(\sum_{c=1}^{N_c}\underline{\ell}^c\otimes\underline{\mathbf{F}}^c\right)-p^\ell\boldsymbol{\delta} \tag{20}$$

which is the same relation as for the fully saturated case. Hence, it follows that the macroscopic strength criterion for the fully saturated situation applies for the high-saturation case providing that the liquids are the same in both situations. Furthermore, the results obtained concerning the effective stress formulation for the macroscopic strength criterion according to the properties of the strength capacities at the contact points remain valid.

Low-Saturation Ratio

It assumed in the sequel that the liquid phase is distributed in disjointed mesniscii located around the contact point or between close grains as for the experiments reported above. For the sake of simplicity, it is also assumed that the granular medium is made up of spherical grains of same radius.

Then, it is useful to introduce the capillary force $\underline{\mathbf{R}}_k^c$ exerted by the liquid meniscus located around the contact c on the grain k

$$\underline{\mathbf{R}}_k^c=\int_{\omega_{k^c}^{s\ell}}(p^g-p^\ell)\underline{\mathbf{n}}\ \, dS+\int_{\partial\omega_{k^c}^{s\ell g}}\gamma^{\ell g}\underline{\boldsymbol{\nu}}\ \, ds \tag{21}$$

One will note that, as long as the effects of gravity can be neglected, the force $\underline{\mathbf{R}}^c$ is equal to the force measured in the experiments reported above.

In order to compute the value of the macroscopic stress tensor $\mathbf{\Sigma}$ associated with a statically admissible microscopic internal force field, consider now the material system made up of one grain, of the solid-gas interface which is located at its boundary and of halves of the capillary meniscii located around the contact points belonging to the considered grain. As this material system complies with a no-body force equilibrium requirement, its contribution to the macroscopic stress tensor $\mathbf{\Sigma}$ writes

$$\sum_{c=1}^{N_c^k}\tilde{\underline{\mathbf{x}}}_k^c\otimes(\underline{\mathbf{F}}_k^c+\underline{\mathbf{R}}_k^c)-v_k p^g\boldsymbol{\delta} \tag{22}$$

where v_k denotes the volume of the material system associated with grain k.

Then, performing the same computation that for the general unsaturated case yields the link between microscopic and macroscopic internal forces suited to the particular case considered here

$$\mathbf{\Sigma}=\frac{1}{|\mathcal{C}|}\sum_{c=1}^{N_c}\underline{\ell}^c\otimes(\underline{\mathbf{F}}^c+\underline{\mathbf{R}}^c)-p^g\boldsymbol{\delta} \tag{23}$$

where $\underline{\mathbf{R}}^c$=capillary force exerted by the liquid meniscus located around the contact c on the grain number k_1^c.

It follows at once that the macroscopic strength criterion writes:

$$\mathcal{G}(p^\ell,p^g)=\mathcal{G}_{\text{cap}}-p^g\boldsymbol{\delta} \tag{24}$$

where $\mathcal{G}_{\text{cap}}$=macroscopic strength criterion of the same granular material in the dry situation which local strength criterion is defined by the set $G_{\text{cap}}^c=G^c+\underline{\mathbf{R}}^c$.

Then, if the local strength criterion is affected neither by the nature of the fluid nor by the value of the fluid pressure, Eq. (24) means that adding meniscii of liquid within a dry granular material increases its macroscopic strength capacities as if the grains were stuck together with a normal compressive force equal to $|\underline{\mathbf{R}}^c|$. This result, which proves to be quite natural, is justified within a complete theoretical background.

Furthermore, it is still possible to use the methods developed in order to solve the yield design problem for the dry granular material to determine the strength criterion in this case.

J. Eng. Mech., 2002, 128(8): 856-863

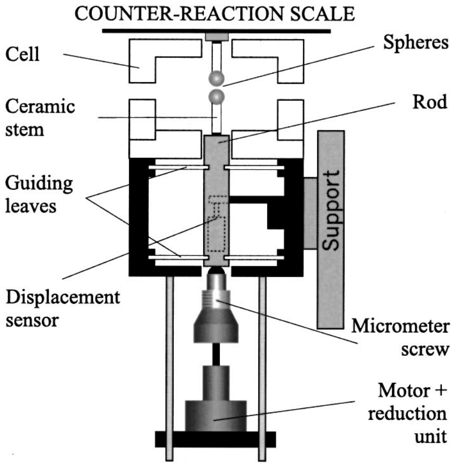

Fig. 1. Experimental liquid bridge apparatus

Liquid Bridge between Two Moving Solid Surfaces

As it does not yet exist, general agreement about one should report capillary effects in a mechanical approach (Biarez et al. 1989; Chateau and Dormieux 1995), some simple experimental results are now presented in order to clarify the situation. Furthermore, the results of these experiments are compared to the theoretical predictions obtained by means of the model used to describe the unsaturated granular material at the microscopic scale in the up-scaling approach. On one hand, different relations enabling to estimate the capillary force $\underline{\mathbf{R}}^c$ of Eqs. (22) and (23) are proposed. Besides, the relevancy of the model used to describe the unsaturated granular medium at the microscopic scale is checked, and there, the validity of the up-scaling approach performed in this paper.

Let us now briefly describe the experimental device. An apparatus dedicated to liquid bridge force measurements has been constructed. Within the experiments reported here, an amount of liquid is injected between two smooth ruby spheres of radius $R = 4$ mm (see Fig. 1). The upper sphere is bolted under the beam of a high-stiffness precision laboratory scale, so that the force F_{cap} exerted by the liquid bridge on the upper sphere can be measured. The gap between the spheres D, as well as the separation velocity, can be adjusted by means of a motor-driven differential micrometer screw. A displacement sensor allow us to obtain the curve $F_{\mathrm{cap}} = f(D)$ with an accuracy of 10 μN for the force and 0.25 μm for the separation distance. Images of the contact region before and after the formation of the meniscus allow the determination of the bridge volume with a precision of about 5%. The whole equipment is inserted in a large thermostated cell. Further and complementary details concerning the experimental procedure performed in order to obtain the results presented below, can be found in (Pitois 1999; Pitois et al. 2000).

Thanks to the symmetry of the solid surfaces, the liquid occupies an axisymmetric domain that can be defined only by its radial profile. The vertical force exerted by the liquid meniscus on the upper sphere is the sum of the vertical component of surface tension forces acting along the wetted perimeter (three-phase contact line) and the pressure force exerted by the liquid over the wetted area.

$$F_{\mathrm{cap}} = p^c \pi R^2 \sin^2\phi + 2\pi R\gamma \sin\phi \sin(\theta+\phi) \qquad (25)$$

where γ = liquid-air interface surface tension; θ = solid-liquid contact angle; ϕ = filling angle defined on the higher sphere; and $p^c = p^g - p^\ell$ is the capillary pressure defined as the pressure jump across the liquid-air interface.

It can be noticed that the term of surface tension forces is always attractive whereas the second depends on the mean curvature of the interface. It can be shown that contribution of the first one can be neglected in some cases (small amounts of liquid for example) but in some other cases its omission leads to repulsive instead of attractive interaction. In contrast with other approaches (Biarez et al. 1989) the two contributions are taken into account in the sequel.

First of all, one is more particularly interested in the measurement of the evolution of the force when the distance between solid surfaces increases or decreases at a rather slow constant speed so that the effects of viscosity are negligible compared to the static capillary forces and this for various values of the volume of liquid (PDMS oil here). For this situation, the difference in hydrostatic pressure across the liquid-air interface is related to the local mean curvature and to the surface tension by the Laplace equation.

The results of two tests are represented on Fig. 2 for a small volume of liquid ($V/R^3 = 0.017$) and on Fig. 3 for a large volume of liquid ($V/R^3 = 0.117$).

For both cases, the attractive capillary force is a decreasing function of the sphere separation distance until the bridge breaks.

The values given by three different theoretical approaches, namely, the exact approach, the toroidal approximation, and the cylindrical approximation, are plotted on the graph.

The results for the exact approach are obtained through the numerical resolution of the Laplace equation completed by the boundary conditions provided by the solid-liquid angle along the triple-line where the liquid-profile interface intersects the two spheres (Pitois et al. 2000). With this numerical solution at hand, it is possible to compute F_{cap} and the volume of the liquid domain binding the two spheres as a function of the filling angle, the surface tension, the liquid weight, and the distance between spheres.

To avoid the use of a numerical method to compute the capillary force, it is possible to approximate the radial profile bounding the liquid domain by a circle. This defines the toroidal approximation. By using such an approximation, it is clear that gravity effect cannot be taken into account. As the tore surface is not a constant-curvature surface, several ways exist in order to calculate the force of interaction F_{cap}, depending upon the profile's point chosen to evaluate it. It can be shown (Pitois et al. 2000) that, referring to the exact method, the best results are obtained by evaluating the force at the gorge of the interface profile. Then a closed-form expression for the force yields. The liquid-domain volume can be easily computed too.

In the limit of small liquid volumes and small gaps and assuming a cylindrical liquid volume (flat profile), a simple closed-form expression (cylindrical expression) can be obtained (Pitois et al. 2000)

$$F_{\mathrm{cap}} = 2\pi R\gamma\cos\theta\left(1 - \frac{1}{\sqrt{1+\dfrac{2V}{\pi R D^2}}}\right) \qquad (26)$$

Comparisons of experimental results with theoretical methods are presented in Figs. 2 and 3 for two liquid bridge volumes. With respect to approximate expressions, it can be noticed that good

J. Eng. Mech., 2002, 128(8): 856-863

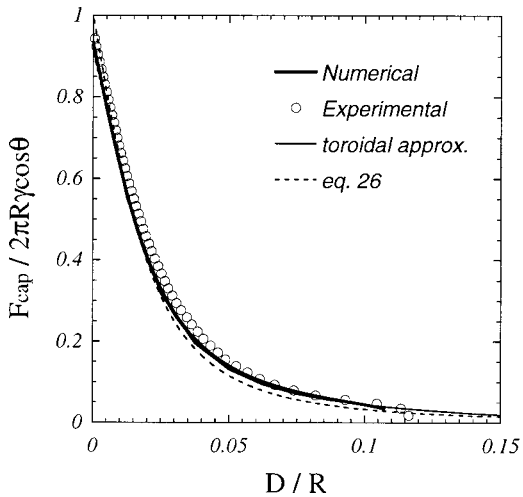

Fig. 2. Experimental and theoretical results for a small volume of liquid ($V/R^3 = 0.001$)

accordance is obtained for the small liquid volume, partly due to the fact that gravity is not taken into account. Besides, although the cylindrical approximation overestimates exact calculations and experimental results, it has been shown elsewhere (Pitois et al. 2000) that expression (26) provides a reasonable approximation for estimating the average force. For the exact method, the small discrepancies of the numerical evaluation can perhaps stem from some inaccuracies in the volume measurement of the liquid bridge. One has to note that the theoretical results are directly computed from the measured volume and liquid-solid contact angle and that no attempt to fit the parameters has been done.

It can be seen that for the two cases, the maximum force is reached when the separation distance is naught.

From a practical point of view, it must be noted that Eq. (26) is sufficiently accurate to be used in order to evaluate the force as a function of the meniscus volume and of the liquid-solid contact angle. Nevertheless, putting $D=0$ in Eq. (26) yields an expression of the maximum force which is not dependent upon the volume of the liquid bridge which is clearly an inexact result.

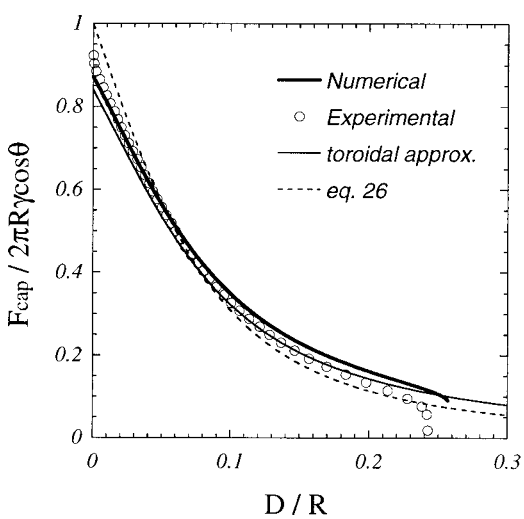

Fig. 3. Experimental and theoretical results for a small volume of liquid ($V/R^3 = 0.017$)

Conclusion

The yield design homogenization method provides a general framework to study the properties of the macroscopic strength criterion from the knowledge of the materials and of the morphology of the media at the microscopic scale.

First of all it has been recalled how to incorporate in this framework an appropriate discrete model to describe the statics of dry granular material. This approach makes it possible to justify that the macroscopic strength criterion of a granular material made up of grains in unilateral contact with dry friction is a cone whose apex is the origin in the space of symmetric stress tensor. The fully saturated case was studied according to the classical approach developed in the framework of saturated porous media and the validity of the Therzaghi effective stress concept to express the strength condition of saturated granular media is assessed.

For the unsaturated situation, experimental results concerning the behavior of a liquid bridge strained between two spheres are given and compared with theoretical predictions. It follows from this comparison that both the pressure and the surface tension must be taken into account in order to accurately compute the force exerted by a liquid meniscus on a grain. Then, the yield design homogenization method is applied to the granular material whose porous space is filled by two fluid phases. In particular, it is shown that the determination of the strength criterion reduces to solving a yield design boundary-value problem defined over the grains occupying the representative cell. Two different morphologies of the fluid phases are studied. When the gas phase (which is also the nonwetting one) occupies small domains surrounded by the liquid phase, the strength capacities are described by the same criterion as the fully saturated granular material. The liquid phase may also be distributed in separate meniscii linking two grains. In this case, the determination of the strength criterion reduces to determine the strength criterion for a dry granular material provided that the local strength criterion has been modified.

As these two morphologies correspond, respectively, to the high-saturation ratio and to the low-saturation ratio, the middle situation, when both the liquid and the gas phases occupy continuous domain remain to be studied. It must be noticed that the main difficulty that remains to be overcomed to perform the complete analysis is the resolution of Laplace equation to determine the shape of the liquid-gas interface for the middle saturation ratio. Nevertheless, for a 2D granular material, these results allow us to determine the strength criterion for all value of the saturation ratio because the case where both fluid phases are continuous does not exist (Urso et al. 1999).

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References

Auriault, J., and Sanchez-Palencia, E. (1977). "Etude du comportement macroscopique d'un milieu poreux saturé déformable." *J. Mécanique*, 16(4), 575–603 (in French).

Biarez, J., Fleureau, J. M., Indarto, Taibi, S., and Zerhouni, M. I. (1989). "Influence of water negative pore pressure on the flow of granular materials in silos." *Powders and grains*, Biarez and Gourvès, eds., Balkema, Rotterdam, The Netherlands, 385–392.

Bourada-Benyamina, N. (1999). ''Etude du comportement des milieux granulaires par homogénéisation périodique.'' Thèse de Doctorat, E.N.P.C., Marne la Vallée (in French).

Caillerie, D. (1995). ''Evolution quasistatique d'un milieu granulaire, loi incrémentale par homogénéisation.'' *Des géomatériaux aux ouvrages—expérimentations et modélisations*, G. Petit, and J. Reynouard, eds., Hermes, Paris, 53–80 (in French).

Cambou, B., Chaze, M., and Dedecker, F. (2000). ''Change of scale in granular materials.'' *Eur. J. Mech. A/Solids,* 19, 999–1014.

Cambou, B., and Sidoroff, F. (1985). ''Description de l'état d'un matériau granulaire par variables internes statiques à partir d'une approche discrète.'' *J.M.T.A.,* 4(2), 223–242 (in French).

Chateau, X., and Dormieux, L. (1995). ''Homogénéisation d'un milieu poreux non saturé: lemme de Hill et applications.'' *C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron.,* 320, 627–634 (in French with an English abridged version).

Christofferson, J., Mehrabadi, M. M., and Nasser, S. N. (1981). ''A micromechanical description of granular material behavior.'' *J. Appl. Mech.,* 48, 339–344.

de Buhan, P. (1986). ''Approche fondamentale du calcul à la rupture des ouvrages en sols renforcés.'' Thèse de Doctorat ès Sc., Université P. et M. Curie, Paris (in French).

de Buhan, P., and Dormieux, L. (1996). ''On the validity of the effective stress concept for assessing the strength of satured porous materials: a homogenization approach.'' *J. Mech. Phys. Solids,* 44(10), 1649–1667.

Drescher, A., and de Josselin de Jong, G. (1972). ''Photoelastic verification of a mechanical model for the flow of a granular material.'' *J. Mech. Phys. Solids*, 20, 337–351.

Emeriault, F., and Cambou, B. (1996). ''Micromechanical modelling of anisotropic non-linear elasticity of granular medium.'' *Int. J. Solids Struct.,* 33(18), 2591–2607.

Fredlund, D. G., and Rahardjo, H. (1993). *Soils mechanics for unsaturated soils*. Wiley, New York.

Hicher, P. Y., and Rahma, A. (1994). ''Micro-macro correlations for granular media. Application to the modelling of sand.'' *Eur. J. Mech. A/Solids*, 13(6), 763–781.

Marshall, T., and Holmes, J. (1988). *Soil physics*, 2nd Ed., Cambridge Univ. Press, Cambridge, U.K.

Pitois, O. (1999). ''Assemblées de grains lubrifiés: Elaboration d'un système modèle expérimental et étude de la loi de contact.'' Thèse de Doctorat, E.N.P.C., Marne la Vallée (in French).

Pitois, O., Moucheront, P., and Chateau, X. (2000). ''Liquid bridge between two moving spheres: an experimental study of viscosity effects.'' *J. Colloid Interface Sci.,* 231, 26–31.

Sab, K. (1996). ''Déformations microscopiques et macroscopiques dans un assemblage dense de particules rigides.'' *C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron.,* 322, 715–721 (in French with an English abridged version).

Salençon, J. (1990). ''An introduction to the yield design theory and its applications to soil mechanics.'' *Eur. J. Mech. A/Solids,* 9(5), 477–500.

Suquet, P. (1982). ''Plasticité et homogénéisation.'' Thèse de Doctorat ès Sc., Université P. et M. Curie, Paris (in French).

Urso, M. E. D., Lawrence, C. J., and Adams, M. J. (1999). ''Pendular, funicular, and capillary bridges: Results for two dimensions.'' *J. Colloid Interface Sci.,* 220, 42–56.

Weber, J. (1966). ''Recherche concernant les contraintes intergranulaires dans les milieux pulvérulents.'' *Bulletin de liaison des ponts et chaussées,* 20, 1–20 (in French).

Zhuang, X., Didwania, A. K., and Goddard, J. D. (1995). ''Simulation of the quasi-static mechanics and scalar transport properties of ideal granular assemblages.'' *J. Comp. Physiol.,* 121, 331–346.

J. Eng. Mech., 2002, 128(8): 856-863